Analysis of Internal Loading at Multiple Robotic Systems

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When multiple robotics systems with several sub-chains grasp a common object, the inherent force redundancy provides a chance of utilizing internal loading. Analysis of grasping space based internal loading is proposed in this work since this method facilitates understanding the physical meaning of internal loadings in some applications, as compared to usual operational space based approach. Investigation of the internal loading for a triple manipulator has been few as compared to a dual manipulator. In this paper, types of the internal loading for dual and triple manipulator systems are investigated by using the reduced row echelon method to analyze the null space of those systems. No internal loading condition is derived and several load distribution schemes are compared through simulation. Furthermore, it is shown that the proposed scheme based on grasping space is applicable to analysis of special cases such as three-fingered and three-legged robots having a point contact with the grasped object or ground.

Key Words: Multiple Robotics Systems, Internal Loading, Load Distribution

1. Introduction

Analysis of an internal loading at multiple robotic systems has been a hot research area. Albert et al. (1988) proposed an weighting matrix in order to minimize unwanted moment at the grasping space. Kumar et al. (1988) defined the internal loading as an interaction force at the walking vehicle and the multi-fingered robot, and proposed no interaction force condition. Lipkin et al. (1991) defined the physical meaning of internal loading using the concept of wrench and twist. Nakamura et al. (1987) defined an internal loading using virtual work principle and optimized the internal loading using the condition of the static friction constraint. Cheng et al. (1991) addressed configuration of two closed chains at the multiple robotics system and force balance at the contact point. Uchiyama et al. (1988) showed the type of the internal loading for the two-arm. Nahon et al. (1992) used the weighting matrix in order to unify the units of force and moment at the algorithm minimizing the internal loading. He also minimized internal loading using quadratic form and two constraint conditions. Choi et al. (1995) proposed minimized constraint condition by a quadratic form and optimized force distribution using a minimized internal loading, but does not describe type of the exact internal loading. Zuo et al. (1999) addressed the difference between an internal force and an interaction force. So, the internal force consists of the interaction and a parallel force at the contact point. Kerr et al. (1989) described the internal force as the grasping force at the multi-fingered hands and proposed optimal selection of an internal grasp force using linear programming. Walker et al.

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(1989) proposed a weighted pseudo-inverse solution that removes an internal loading in particular solution. Yoshikawa et al.(1987, 1999) defined a grasping force that is the internal force satisfying the friction constraint and manipulating force, and also proposed an virtual truss as a model of the object grasped at n contact points. Li et al.(2003) proposed an algorithm for three-finger force-closure grasp.

Based on literature survey, the internal loading in multiple arms has been analyzed using the operational space approach except few cases. However, the notion of internal loading defined in the operational space is not visible, because it is explained at a single operational position. This paper claims that in some application, grasping space approach has advantage to represent the physical meaning of internal loading, in comparison to operational space approach. One example is the case of an elastic object that is grasped by many robots. Another case is the multi-fingered hands or walking machine, in which the interaction load between two adjacent fingers or legs can be interpreted as an example of grasping force.

Linear algebraically, it is known that the particular solution corresponding to the motion part does not create any internal loading, and only the homogeneous part has something to do with internal loading. The Walker's algorithm (Walker, 1989), defined in the operational space, claims that the internal loading also exists in the particular solution in multiple arm operation. And thus they proposed a way to eliminate a probable internal loading in the particular solution. This paper shows that the typical pseudo-inverse solution does not in fact create any internal loading. On the other hand, the weighted pseudo-inverse solution proposed by Walker actually created internal loading. This fact was shown through a simple example.

In this paper, we attempt to analyze the internal loading at multiple robotic systems by using reduced row echelon method and show the shape of the basis of the internal loading. Grasping space based analysis will be performed to provide better physical meaning of the internal loading. Secondly, we derive the condition of no internal loading and compare the internal loading defined at the operational space with that defined at the grasping space for dual and triple robotic systems. Lastly, using the same methodology, three-fingered and three-legged systems are illustrated as special cases.

2. Concept and Type of Internal Loading

2.1 Concept of the internal loading

An invisible force and moment are exerted on the grasped object by each manipulator at the multiple robotic systems. In this paper, we define the internal loading as the forces and moments that does not affect the motion of the end-effector. The relation between the total forces (f)/moments (m) at the grasping space $(\underline{F})$ with n number of manipulators and forces/moments at the operational space $(\underline{P})$ can be described as [1]

$$\underline{P}=G\underline{F} \tag{1}$$

where

$$\underline{P}=[f_x\ f_y\ f_z\ m_x\ m_y\ m_z]^T \tag{2}$$

$$\underline{F}=[\underline{F_1}^T\ \underline{F_2}^T\ \mathrm{L}\ \underline{F_n}^T]^T \tag{3}$$

with

$$\underline{F_i}=[f_{xi}\ f_{yi}\ f_{zi}\ m_{xi}\ m_{yi}\ m_{zi}]^T \quad \text{for } i=1,2,\cdots,n \tag{4}$$

and the transformation matrix can be expressed as

$$G=\begin{bmatrix} I_3 & 0 & I_3 & 0 & \cdots & I_3 & 0 \\ S_1 & I_3 & S_2 & I_3 & \cdots & S_n & I_3 \end{bmatrix} \tag{5}$$

with skew symmetric matrix

$$S_i=\begin{bmatrix} 0 & -r_{z_i} & r_{y_i} \\ r_{z_i} & 0 & -r_{x_i} \\ -r_{y_i} & r_{x_i} & 0 \end{bmatrix} \tag{6}$$

for the position vector of the end-effector of the i-th manipulator, $\underline{r_i}=[r_{x_i}\ r_{y_i}\ r_{z_i}]^T$, and 3 by 3 identity matrix, I_3, respectively.

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1 Grasping space implies the space where manipulators grasp an object, and operational space implies the output space.

The general solution of Eq. (1) can be expressed as

$$\underline{F}=G_W^+\underline{P}+(I-G_W^+G)\underline{\varepsilon} \tag{7}$$

where, when $\underline{G}$ is row full-rank, the weighted pseudo-inverse with weighting matrix, W, can be expressed as

$$G_W^+=W^{-1}G^T(GW^{-1}G^T)^{-1} \tag{8}$$

and $\underline{\varepsilon}$ denotes an arbitrary vector. Doty et al. (1993) addressed the weighted pseudo-inverse for more general cases. The first term on the right-hand side of Eq. (7) represents a particular solution, and the second term denotes a homogenous solution that creates internal loading without affecting the motion involved in the particular solution.

When we map the force and moment from the grasping space to the operational (or object) space, Walker et al. (1989) claimed that the particular solution has additional internal loading at the operational space. Based on his observation, he proposed no squeezing weighted pseudo-inverse that removed an internal loading at the operational space.

Suppose that $\underline{\bar{F}}_i$ denotes the effective forces and moments of the i-th manipulator at the operational space. Then, the transformation is given by

$$\underline{\bar{F}}_i=\bar{G}_i\underline{F}_i \tag{9}$$

where the effective transformation matrix for i-th manipulator is given by

$$\bar{G}_i=\begin{bmatrix} I_3 & 0 \\ S_i & I_3 \end{bmatrix} \tag{10}$$

Let

$$\Delta=\begin{bmatrix} 0 & kI_3 & & & & & & 0 \\ kI_3 & 0 & & & & & & \\ & & 0 & kI_3 & & & & \\ & & kI_3 & 0 & & & & \\ & & & & O & & & \\ & & & & & & 0 & kI_3 \\ 0 & & & & & & kI_3 & 0 \end{bmatrix} \tag{11}$$

where $\Delta\in\Re^{6n\times 6n}$ and k is non-zero scalar. When $W=\Delta$, G_Δ^+ in Eq. (8) is given by (Walker, 1989)

$$G_\Delta^+=\Delta G^T(G\Delta G^T)^{-1} \tag{12}$$

Now, Eq. (12) also can be expressed as

$$G_\Delta^+=\Delta G^T(G\Delta G^T)^{-1}=\frac{1}{n}\begin{bmatrix} I_3 & 0 \\ -S_1 & I_3 \\ \vdots & \vdots \\ I_3 & 0 \\ -S_n & I_3 \end{bmatrix} \tag{13}$$

where $G_\Delta^+=\Delta G^T(G\Delta G^T)^{-1}$ denotes no squeezing pseudo-inverse suggested by Walker et al. (1989). Accordingly, the internal loading term given in Eq. (5) also will be replaced by $(I-G_\Delta^+G)$.

In this paper, the geometry of the internal loading terms will be analyzed in detail. The claim of Walker is opposite to previous concept of internal loading, which exists only in the homogeneous solution. This paper reclaims that the Walker algorithm is incorrect and that the previous algorithm is correct by showing proof and some illustrative examples.

2.2 Types of internal loading

2.2.1 Two manipulators in the 2-dimensional space

The concept of internal loading will be explained with a simple example given in Fig. 1. XYZ denotes a coordinate frame attached to the object. Assume that the two manipulators are at the same distance from XYZ coordinate, and grasp the common object. In the plane, the transformation matrix of Eq. (5) can be described as

$$G=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -r_{y1} & r_{x1} & 1 & -r_{y2} & r_{x2} & 1 \end{bmatrix} \tag{14}$$

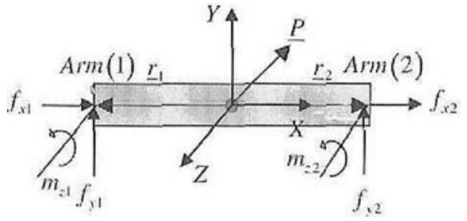


Fig. 1 Planar motion by two manipulators

and $\underline{P}$ and $\underline{F}$ are given by

$$\underline{P}=[f_x\ f_y\ m_z]^T \tag{15}$$

$$\underline{F}=[\underline{F}_1^T\ \underline{F}_2^T]^T \tag{16}$$

with

$$\underline{F}_i=[f_{xi}\ f_{yi}\ m_{zi}]^T \text{ for } i=1,\ 2 \tag{17}$$

respectively. When the position vectors are given by $\underline{r}_1=[-1\ \ 0]^T$ and $\underline{r}_2=[1\ \ 0]^T$, the internal loading matrix of this case can be obtained as

$$I-G^+G=\begin{bmatrix} 0.5 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0.25 & 0.25 & 0 & -0.25 & 0.25 \\ 0 & 0.25 & 0.75 & 0 & -0.25 & -0.25 \\ -0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & -0.25 & -0.25 & 0 & 0.25 & -0.25 \\ 0 & 0.25 & -0.25 & 0 & -0.25 & 0.75 \end{bmatrix} \tag{18}$$

where G^+ denotes the weighted pseudo inverse with identity weight. This matrix is a 6 by 6 square matrix whose rank is 3. Thus, this matrix has three independent internal loadings. So, we can obtain the basis of internal loading from Eq. (18) by using row reduced echelon method. The basis of the internal loading in the planar case can be obtained as the following the three vectors

$$\underline{F}=[f_{x1}\ f_{y1}\ m_{z1}\ f_{x2}\ f_{y2}\ m_{z2}]^T$$

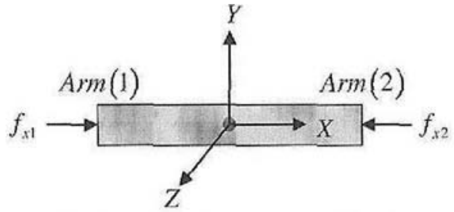
$$[1\ 0\ 0\ -1\ 0\ 0]^T: x-force \tag{19}$$

$$[0\ 1\ 2\ 0\ -1\ 0]^T: y-force \tag{20}$$

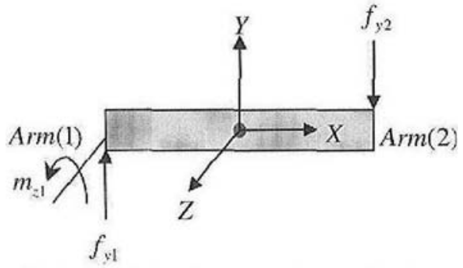
$$[0\ 0\ -1\ 0\ 0\ 1]^T: z-moment \tag{21}$$

Fig. 2 shows the shape of the internal loading given by Eqs. (19)-(21). The X-directional internal force and Z-directional internal moment act on the opposite direction to each other. However, in the Y-direction, Z-directional moment should be applied to sustain equilibrium for a given pair of the opposite Y-directional forces. Physically, this represents a situation that the left end of the object is clamped while sustaining equilibrium.

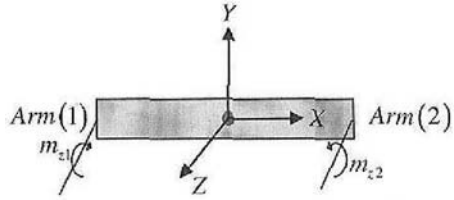
If the object is not a rigid body and a compliant body, it will be deformed by applying these internal forces and moments.



(a) Internal loading in the X-direction



(b) Internal loading in at the Y-direction



(c) Internal moment in the Z-direction

Fig. 2 Internal loading by two manipulators for planar case

We only analyzed the homogenous solution term. Now, we will analyze the particular solution term. The particular solution generally does not affect the internal loading, which is opposite to Walker's algorithm. So, we would like to clarify this argument by simulation. The relationship between the force and moment at the operational space and the grasping space can be respectively defined by transformation matrix.

$$\underline{\bar{F}}_1=\bar{G}_1\underline{F}_1,\ \underline{\bar{F}}_2=\bar{G}_2\underline{F}_2 \tag{22}$$

When the planar dual arm of Fig. 1 moves in the positive Y-direction, the operational force can be expressed as $\underline{P}=(0\ 1\ 0)^T$. For this case, Table 1 shows the difference of the force and moment between the grasping space and operational space. This result shows that the internal loading exists at the operational space, while the internal loading does not exist in the grasping space as shown

Table 1 Comparison of internal loadings

Space Region	$\underline{P}$	[0 1 0]
The operational space	$\underline{F}_1$	[0 0.5 −0.5]
	$\underline{F}_2$	[0 0.5 0.5]
The grasping space	$\underline{F}_1$	[0 0.5 0]
	$\underline{F}_2$	[0 0.5 0]

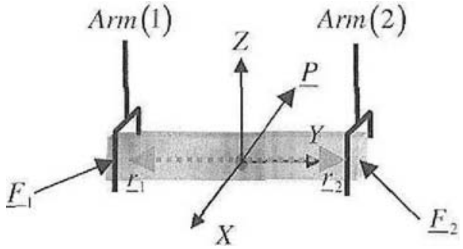


Fig. 3 Spatial motion by two manipulators

in Fig. 2. Actually, the result based on the grasping space makes more sense because it is physically correct, but the internal loading (0.5, −0.5) based on the operational space is fallacious because it is not actually internal force. Therefore, in this work, we will analyze the internal loading at the grasping space.

2.2.2 Two manipulators in the spatial domain

Now, we consider a spatial case and assume that two manipulators grasp a common object in the spatial domain. $\underline{P}$, $\underline{F}$ and G can be described as

$$\underline{P}=[f_x\ f_y\ f_z\ m_x\ m_y\ m_z]^T \tag{23}$$

$$\underline{F}=[\underline{F}_1^T\ \underline{F}_2^T]^T \tag{24}$$

with

$$\underline{F}_i=[f_{xi}\ f_{yi}\ f_{zi}\ m_{xi}\ m_{yi}\ m_{zi}]^T \text{ for } i=1,\ 2 \tag{25}$$

$$G=\begin{bmatrix} I_3 & 0 & I_3 & 0 \\ S_1 & I_3 & S_2 & I_3 \end{bmatrix} \tag{26}$$

respectively. When the position vectors are located at $\underline{r}_1=[0\ \ -1\ \ 0]^T$ and $\underline{r}_2=[0\ \ 1\ \ 0]^T$, the internal loading matrix $(I-G^+G)$ can be obtained as

$$\begin{bmatrix}
0.25 & 0 & 0 & 0 & 0 & -0.25 & -0.25 & 0 & 0 & 0 & 0 & -0.25 \\
0 & 0.5 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 & 0 & -0.25 & 0.25 & 0 & 0 \\
0 & 0 & 0.25 & 0.75 & 0 & 0 & 0 & 0 & -0.25 & -0.25 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 \\
-0.25 & 0 & 0 & 0 & 0 & 0.75 & 0.25 & 0 & 0 & 0 & 0 & -0.25 \\
-0.25 & 0 & 0 & 0 & 0 & 0.25 & 0.25 & 0 & 0 & 0 & 0 & 0.25 \\
0 & -0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.25 & -0.25 & 0 & 0 & 0 & 0 & 0.25 & -0.25 & 0 & 0 \\
0 & 0 & 0.25 & -0.25 & 0 & 0 & 0 & 0 & -0.25 & 0.75 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \\
-0.25 & 0 & 0 & 0 & 0 & -0.25 & 0.25 & 0 & 0 & 0 & 0 & 0.75
\end{bmatrix} \tag{27}$$

This matrix is a 12 by 12 square matrix whose rank is 6. Therefore, there are six independent internal loadings. We find the basis of the internal loading from Eq. (27) by using reduced row echelon method. The basis of the internal loading can be obtained as following

$$\underline{F}=[f_{x1}\ f_{y1}\ f_{z1}\ m_{x1}\ m_{y1}\ m_{z1}\ f_{x2}\ f_{y2}\ f_{z2}\ m_{x2}\ m_{y2}\ m_{z2}]^T$$

$$[-1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 1]^T: x\textit{-force} \tag{28}$$

$$[0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0]^T: y\textit{-force} \tag{29}$$

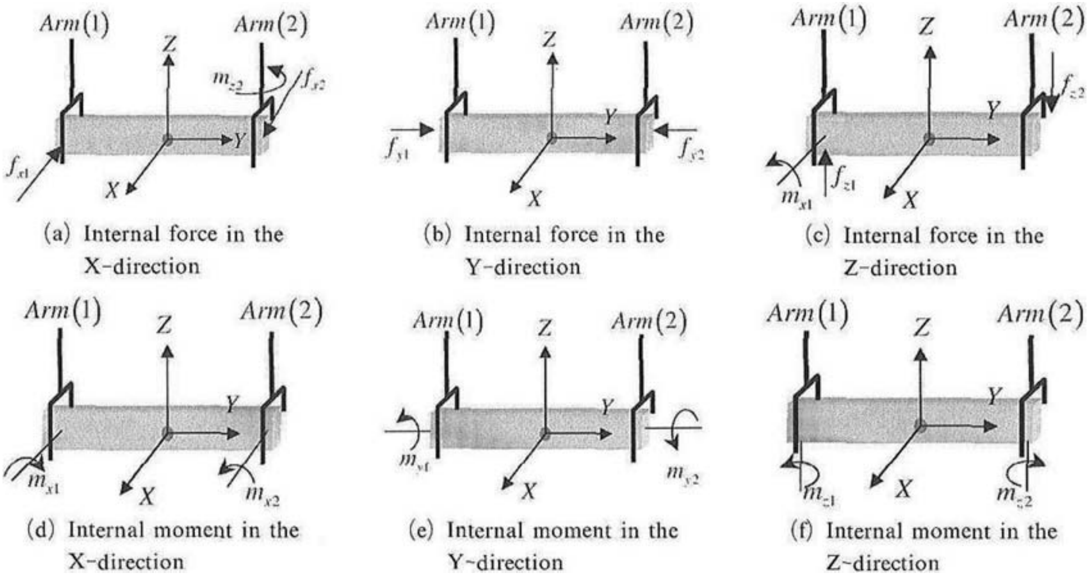
$$[0\ 0\ 1\ 2\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0]^T: z\textit{-force} \tag{30}$$

$$[0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0]^T: x\textit{-moment} \tag{31}$$

$$[0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0]^T: y\textit{-moment} \tag{32}$$

$$[0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1]^T: z\textit{-moment} \tag{33}$$

Fig. 4 shows the shape of the six internal loadings. The X-directional internal forces have the same magnitude, the same line of action, but the opposite direction. Therefore, an offset moment in the Z-direction should be given at the second manipulator in order to sustain equilibrium. The Y-directional forces have the same magnitude, the same line of action, but the opposite direction. The Z-directional forces have the same magni-



(a) Internal force in the X-direction
(b) Internal force in the Y-direction
(c) Internal force in the Z-direction
(d) Internal moment in the X-direction
(e) Internal moment in the Y-direction
(f) Internal moment in the Z-direction

Fig. 4 Internal loading of a dual manipulator in 3-dimensional space

tude, the same line of action, but the opposite sense.

In that case, the moment in the X-direction should be applied at the first manipulator in order to sustain equilibrium. The moments in the X-, Y-, and Z-directions have the same magnitude, but the opposite direction, respectively.

Similar to the planar case, the object is clamped in the X- and Z- directions. Thus, this case has one linear internal loading, two clamped type internal loadings, and three moment type internal loadings.

2.2.3 Triple manipulator in planar domain

When three manipulators rigidly grasp a common object in the planar domain, the position vectors is given as $\underline{r}_1=[-1 \quad -1/\sqrt{3}]^T$, $\underline{r}_2=[1 \quad -1/\sqrt{3}]^T$, and $\underline{r}_3=[0 \quad 2/\sqrt{3}]^T$. The transformation matrix can be expressed as

$$G=\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ -r_{y1} & r_{x1} & 1 & -r_{y2} & r_{x2} & 1 & -r_{y3} & r_{x3} & 1 \end{bmatrix} \quad (34)$$

and $\underline{F}_i$ of the i-th manipulator can be expressed as $[f_{xi} \; f_{yi} \; m_{zi}]^T$, for $i=1,2,3$, Then the internal loading matrix can be obtained as

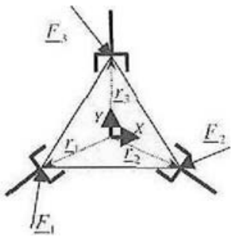


Fig. 5 Triple manipulator in planar domain

$$I-G^{+}G=\begin{bmatrix} 0.62 & 0.08 & -0.08 & -0.38 & -0.08 & -0.08 & -0.24 & 0 & -0.08 \\ 0.08 & 0.52 & 0.14 & 0.08 & -0.19 & 0.14 & -0.17 & -0.33 & 0.14 \\ -0.08 & 0.14 & 0.86 & -0.08 & -0.14 & -0.14 & 0.17 & 0 & -0.14 \\ -0.38 & 0.08 & -0.08 & 0.62 & -0.08 & -0.08 & -0.24 & 0 & -0.08 \\ -0.08 & -0.19 & -0.14 & -0.08 & 0.52 & -0.14 & 0.17 & -0.33 & -0.14 \\ -0.08 & 0.14 & -0.14 & -0.08 & -0.14 & 0.86 & 0.17 & 0 & -0.14 \\ -0.24 & -0.17 & 0.16 & 0.24 & 0.17 & 0.17 & 0.48 & 0 & 0.17 \\ 0 & -0.33 & 0 & 0 & -0.33 & 0 & 0 & 0.67 & 0 \\ -0.08 & 0.14 & -0.14 & -0.08 & -0.14 & -0.14 & 0.17 & 0 & 0.86 \end{bmatrix} \quad (35)$$

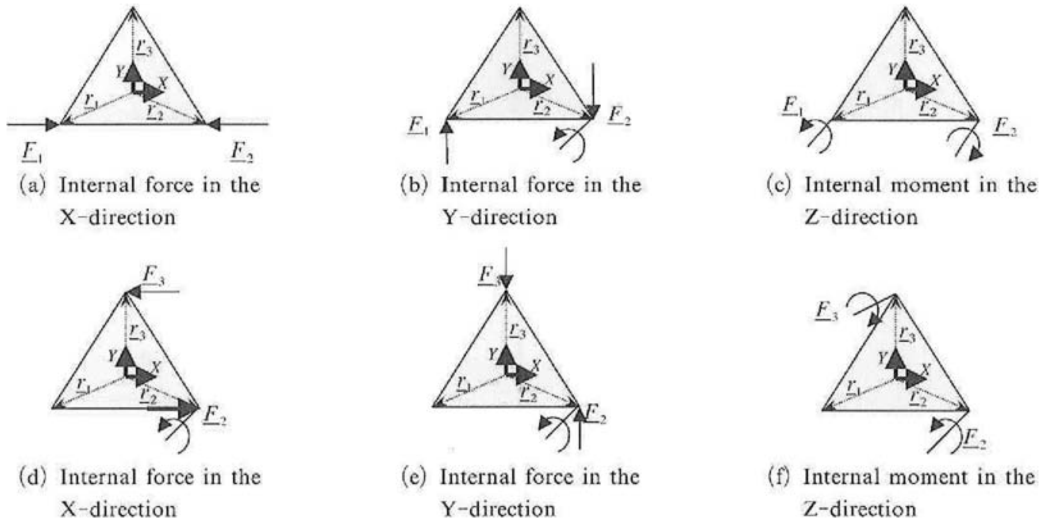


Fig. 6 Shape of the basis of internal loading for triple manipulator in the planar domain

This matrix is a 9 by 9 square matrix whose rank is 6. Therefore, by using the row-reduced echelon method, the dimension of the internal loading basis is 6. The six vectors consisting of the basis can be obtained as

$$F=[f_{x1}f_{y1}m_{z1}\ f_{x2}f_{y2}m_{z2}\ f_{x3}f_{y3}m_{z3}]^T$$

$$[1\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ 0] \quad (36)$$

$$[0\ 1\ 0\ 0\ -1\ 2\ 0\ 0\ 0] \quad (37)$$

$$[0\ 0\ 1\ 0\ 0\ -1\ 0\ 0\ 0]^T \quad (38)$$

$$[0\ 0\ 0\ 1\ 0\ 0\ -1\ 0\ 1.73]^T \quad (39)$$

$$[0\ 0\ 0\ 0\ 1\ 0\ 0\ -1\ 1]^T \quad (40)$$

$$[0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ -1]^T \quad (41)$$

Fig. 6(b), (d), and (e) show the internal loadings that come from the combination of the internal force and moment. If the internal force that is symmetric to the referenced coordinate is exerted on the ($\underline{r}_i-\underline{r}_j$) line, then the moment does not occur. However, the internal forces are generally coupled to internal moment. Fig. 6(c) and (f) show that the internal moment independently occurs because the moment is a free vector. Thus, this triple manipulator has the internal loadings of one linear type, three clamped types, and two pure moment types.

2.2.4 Triple manipulator in spatial domain

When a robotic system consists of three manipulators at three dimensional space, the number of independent internal loading will be 12. Analysis of the internal loading for the triple arm has not been addressed so far because of the complexity of the internal loading. When the position vectors of the end-effector for the three manipulators are given by $\underline{r}_1=[0\ -1\ 0]^T$, $\underline{r}_2=[0\ 1\ 0]^T$, and $\underline{r}_3=[-\sqrt{3}\ 0\ 0]^T$. The G and $\underline{F}$ are respectively given by

$$G=\begin{bmatrix} I_3 & 0 & I_3 & 0 & I_3 & 0 \\ S_1 & I_3 & S_2 & I_3 & S_3 & I_3 \end{bmatrix} \quad (42)$$

$$\underline{F}=[\underline{F}_1^T\ \underline{F}_2^T\ \underline{F}_3^T]^T \quad (43)$$

then the internal loading matrix $(I-G^+G)$ can be obtained as

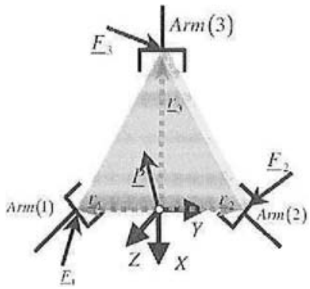
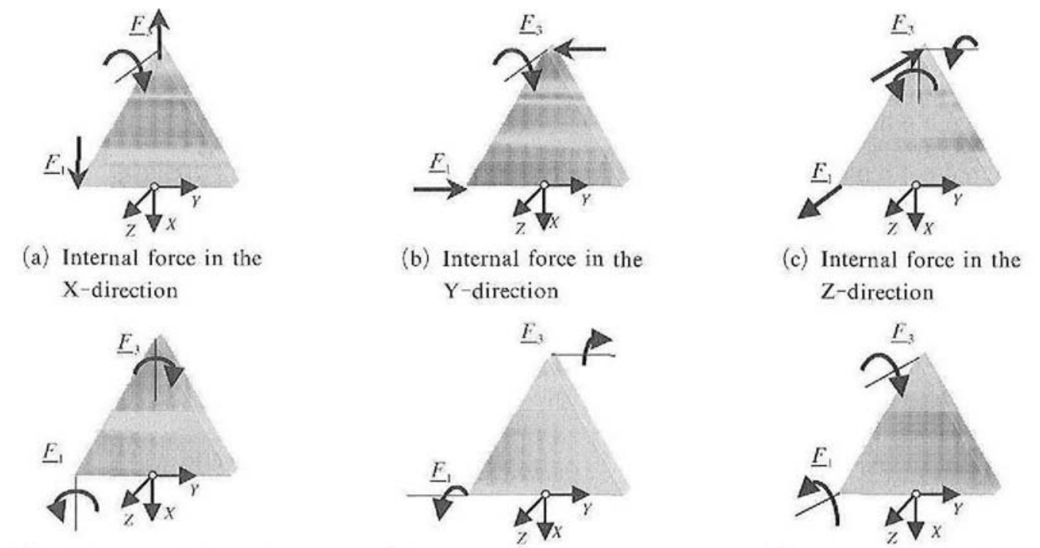


Fig. 7 Triple manipulator in spatial domain



(a) Internal force in the X-direction (b) Internal force in the Y-direction (c) Internal force in the Z-direction

(d) Internal moment in the X-direction (e) Internal moment in the Y-direction (f) Internal moment in the Z-direction

Fig. 8 Shape of internal loading basis for triple manipulator in spatial domain

$$\begin{bmatrix}
0.52 & 0 & 0 & 0 & -0.14 & -0.19 & -0.08 & 0 & 0 & 0 & -0.14 & -0.33 & 0.17 & 0 & 0 & 0 & -0.14 \\
-0.08 & 0 & 0 & 0 & -0.08 & 0.08 & -0.38 & 0 & 0 & 0 & -0.08 & 0 & -0.24 & 0 & 0 & 0 & -0.08 \\
0 & 0.4 & 0.2 & 0.12 & 0 & 0 & 0 & -0.2 & 0.2 & 0.12 & 0 & 0 & 0 & -0.2 & 0.2 & 0.12 & 0 \\
0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 & -0.2 & -0.2 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & 0 \\
0 & 0.12 & 0 & 0.8 & 0 & 0 & 0 & 0.12 & 0 & -0.2 & 0 & 0 & 0 & -0.23 & 0 & -0.2 & 0 \\
-0.14 & 0 & 0 & 0 & 0.86 & 0.14 & -0.08 & 0 & 0 & 0 & -0.14 & 0 & 0.17 & 0 & 0 & 0 & -0.14 \\
-0.19 & 0 & 0 & 0 & 0.14 & 0.53 & 0.08 & 0 & 0 & 0 & 0.14 & -0.33 & -0.17 & 0 & 0 & 0 & 0.14 \\
-0.8 & 0 & 0 & 0 & -0.08 & 0.08 & 0.62 & 0 & 0 & 0 & -0.08 & 0 & -0.24 & 0 & 0 & 0 & -0.08 \\
0 & -0.12 & -0.2 & 0.12 & 0 & 0 & 0 & 0.4 & -0.2 & 0.12 & 0 & 0 & 0 & -0.2 & -0.2 & 0.12 & 0 \\
0 & 0.2 & -0.2 & 0 & 0 & 0 & 0 & -0.2 & 0.8 & 0 & 0 & 0 & 0 & 0 & -0.2 & 0 & 0 \\
0 & 0.12 & 0 & -0.2 & 0 & 0 & 0 & 0.12 & 0 & 0.8 & 0 & 0 & 0 & -0.23 & 0 & -0.2 & 0 \\
-0.14 & 0 & 0 & 0 & -0.14 & 0.14 & 0.08 & 0 & 0 & 0 & 0.86 & 0 & 0.17 & 0 & 0 & 0 & -0.14 \\
-0.33 & 0 & 0 & 0 & 0 & -0.33 & 0 & 0 & 0 & 0 & 0 & 0.67 & 0 & 0 & 0 & 0 & 0 \\
0.17 & 0 & 0 & 0 & 0.17 & -0.17 & -0.24 & 0 & 0 & 0 & 0.17 & 0 & 0.48 & 0 & 0 & 0 & 0.17 \\
0 & -0.2 & 0 & -0.23 & 0 & 0 & 0 & -0.2 & 0 & -0.23 & 0 & 0 & 0 & 0.4 & 0 & -0.23 & 0 \\
0 & 0.2 & -0.2 & 0 & 0 & 0 & 0 & -0.2 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0.8 & 0 & 0 \\
0 & 0.12 & 0 & -0.2 & 0 & 0 & 0 & 0.12 & 0 & -0.2 & 0 & 0 & 0 & -0.23 & 0 & 0.8 & 0 \\
-0.14 & 0 & 0 & 0 & -0.14 & 0.14 & -0.08 & 0 & 0 & 0 & -0.14 & 0 & 0.17 & 0 & 0 & 0 & 0.86
\end{bmatrix} \quad (44)$$

This matrix is a 18 by 18 square matrix whose rank is 12. Therefore, using row-reduced echelon method, the basis of internal loading consists of 12 vectors that can be obtained as

$$[1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ -1]^T \quad (45)$$

$$[0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ -1.03]^T \quad (46)$$

$$[0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 1\ 1.73\ 0]^T \quad (47)$$

$$[0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0]^T \quad (48)$$

$$[0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1\ 0]^T \quad (49)$$

$$[0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ -1]^T \quad (50)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 0\ 1]^T \quad (51)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0\ 0\ 1.73]^T \quad (52)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ -1\ 1.73\ 0]^T \quad (53)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0\ 0]^T \quad (54)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1\ 0]^T \quad (55)$$

$$[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 0\ 0\ -1]^T \quad (56)$$

The trends are similar to the planar case. Internal moments also independently happen. However, internal forces are coupled to internal moments that are called couple moment. Fig. 8 shows the shape of the internal loading between the first and third manipulator. The other internal loading can be also visualized similarly.

2.3 General case

The above analysis can be extended to the general case. The number of the internal loading basis is $6\times(n-1)$ at the spatial motion, $3\times(n-1)$ at the planar motion with n number of manipulators.

3. Condition of no Internal Loading at the Particular Solution

It has been argued whether the particular solution in the force resolution problem of multiple robotic systems has some internal force or not. Walker et al. (1989) suggested a weighted pseudo-inverse solution to resolve this problem. However, we have shown that Walker's algorithm has not been proven as shown in the sub section 2.2.1. It will be shown that a weighted pseudo inverse solution has the additional internal squeezing force.

⟨Proposition⟩

The row space and the null space of Eq. (7) are mutually orthogonal if and only if the weighting matrix is an identity matrix.

⟨proof⟩

Assume that the row space and the null space are mutually orthogonal. Then, the inner product of those will always satisfy

$$G_W^{+T}(I-G_W^{+}G)=0 \qquad (57)$$

Eq. (57) can be expressed, in more detail, as

$$(GW^{-1}G^T)^T GW^{-T} - (GW^{-1}G^T)^{-T}GW^{-T}W^{-1}G^T(GW^{-1}G^T)^{-1}G=0 \qquad (58)$$

Assuming

$$W^{-T}W^{-1}=W^{-1} \qquad (59)$$

then, Eq. (58) can be expressed as

$$(GW^{-1}G^T)^{-T}G(W^{-T}-I)=0 \qquad (60)$$

If $GW^{-1}G^T\neq 0$ in Eq. (60), then $(W^{-T}-I)$ must be zero or G and $(W^{-T}-I)$ should be orthogonal. Both cases yield

$$W^{-T}=I \qquad (61)$$

Taking transpose and inverse on both sides yields

$$W=I \qquad (62)$$

Eq. (62) also satisfies the initial assumption given by Eq. (59).

Now, let's assume that the weighting matrix is given as an identity matrix. Then Eq. (58) can be expressed as

$$(GG^T)^{-T}G-(GG^T)^{-T}GG^T(GG^T)^{-1}G \qquad (63)$$

where, since $GG^T(GG^T)^{-1}=I$

$$(GG^T)^{-T}G-(GG^T)^{-T}G=0 \qquad (64)$$

Thus, it is proven that row space and the null space are mutually orthogonal.

As a conclusion, when the weighting matrix is given as an identity matrix, the particular solution does not have internal loading at the grasping space. However, if the weighting matrix is given arbitrarily, then the internal loading exists at the grasping space. For instance, when the grasping positions are given as $\underline{r_1}=[-1\ 0]^T$ and $\underline{r_2}=[1\ 0]^T$, and the weighting matrix is a 6 by 6 identity matrix. Table 2 shows that all six cases

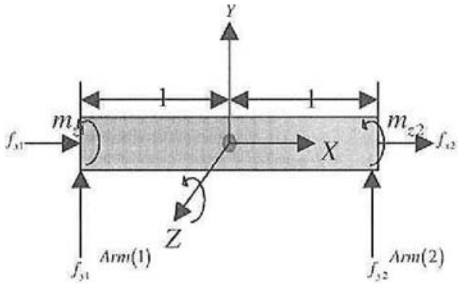


Fig. 9 Planar motion

Table 2 Load distribution for identity weighting

	Case 1	Case 2	Case 3
$\underline{P}$	$[0\ 1\ 1]^T$	$[0\ 0\ 1]^T$	$[1\ 0\ 0]^T$
$\underline{F}_1$	$[0\ 0.25\ 0.25]^T$	$[0\ -0.25\ 0.25]^T$	$[0.5\ 0\ 0]^T$
$\underline{F}_2$	$[0\ 0.75\ 0.25]^T$	$[0\ 0.25\ 0.25]^T$	$[0.5\ 0\ 0]^T$
	Case 4	Case 5	Case 6
$\underline{P}$	$[0\ 1\ 0]^T$	$[1\ 1\ 0]^T$	$[1\ 0\ 1]^T$
$\underline{F}_1$	$[0\ 0.5\ 0]^T$	$[0.5\ 0.5\ 0]^T$	$[0.5\ -0.25\ 0.25]^T$
$\underline{F}_2$	$[0\ 0.5\ 0]^T$	$[0.5\ 0.5\ 0]^T$	$[0.5\ 0.25\ 0.25]^T$

Table 3 Load Distribution for Nonidentity Weighting (planar case)

	Case 1	Case 2	Case 3
$\underline{P}$	$[0\ 1\ 1]^T$	$[0\ 0\ 1]^T$	$[1\ 0\ 0]^T$
$\underline{F}_1$	$[-0.5\ 0.5\ 0.5]^T$	$[0\ 0\ 0.5]^T$	$[0.5\ 0\ 0]^T$
$\underline{F}_2$	$[0.5\ 0.5\ 0.5]^T$	$[0\ 0\ 0.5]^T$	$[0.5\ 0\ 0]^T$
	Case 4	Case 5	Case 6
$\underline{P}$	$[0\ 1\ 0]^T$	$[1\ 1\ 0]^T$	$[1\ 0\ 1]^T$
$\underline{F}_1$	$[-0.5\ 0.5\ 0]^T$	$[0\ 0.5\ 0]^T$	$[0.5\ 0\ 0.5]^T$
$\underline{F}_2$	$[0.5\ 0.5\ 0]^T$	$[1\ 0.5\ 0]^T$	$[0.5\ 0\ 0.5]^T$

do not exhibit any internal loading for the given *operational load* ($\underline{P}$). *Refer to Fig. 2 for three* types of internal loading in this example.

However, when the weighting matrix is given as, which is an arbitrary nonidentity matrix

$$W=\begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0 \end{bmatrix} \tag{65}$$

then the case 1 and 4 of Table 3 yield some internal loading along the X-direction. Fig. 10 (a) and (b) show the case of no internal loading. Fig. 10(c) and (d) show the cases of the internal loading.

Figure 11 shows a dual-arm operated in 3-dimensional space. The grasping position of the two arms is given as $\underline{r}_1=[-1\ 0\ 0]^T$ and $\underline{r}_2=[1\ 0\ 0]^T$, and the weighting matrix is a 12 by 12 identity

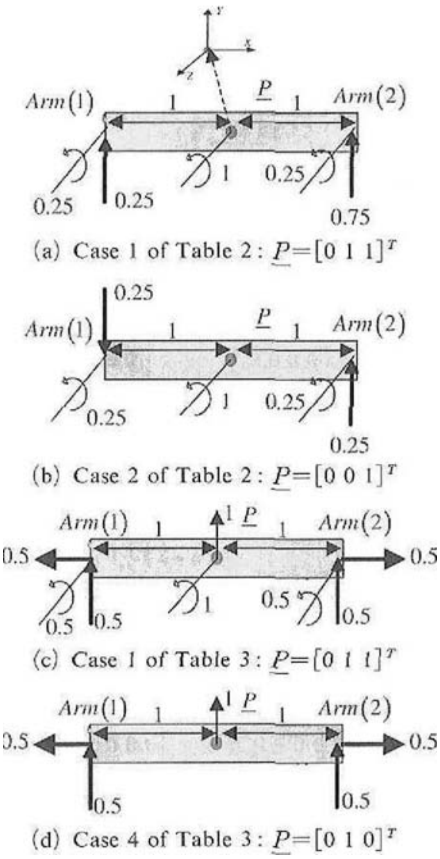


Fig. 10 Comparison of the internal loading of the table 1and 2

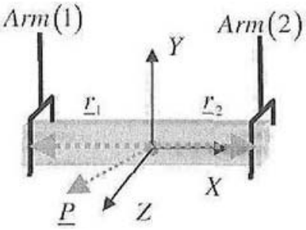


Fig. 11 Spatial motion by two manipulators

matrix. As shown in Table 4, this example has no internal loading.

When we map the force and moment from the grasping space to the operational space, Walker et al.(1989) suggested that the particular solution has additional internal loading at the operational space. Therefore, Walker proposed no squeezing weighted pseudo-inverse that removes the internal loading at the operational space. However, when the weighting matrix

$$W=\begin{bmatrix} W_1 & 0 \\ 0 & W_1 \end{bmatrix} \text{ with } W_1=\begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} \tag{66}$$

Table 4 Loading Distribution in 3-Dimensional Space with Identity Weighing

	Case 1	Case 2
$\underline{P}$	$[0\ 1\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 1\ 0\ 0\ 0]^T$
$\underline{F}_1$	$[0\ 0.5\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 0.5\ 0\ 0\ 0]^T$
$\underline{F}_2$	$[0\ 0.5\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 0.5\ 0\ 0\ 0]^T$
	Case 3	Case 4
$\underline{P}$	$[0\ 1\ 0\ 0\ 1\ 0]^T$	$[0\ 0\ 1\ 0\ 0\ 1]^T$
$\underline{F}_1$	$[0\ 0.5\ 0.25\ 0\ 0.25\ 0]^T$	$[0\ -0.25\ 0.5\ 0\ 0\ 0.25]^T$
$\underline{F}_2$	$[0\ 0.5\ -0.25\ 0\ 0.25\ 0]^T$	$[0\ 0.25\ 0.5\ 0\ 0\ 0.25]^T$

Table 5 Load Distribution in 3-Dimensional Space with Nonidentity Weighting

	Case 1	Case 2
$\underline{P}$	$[0\ 1\ 0\ 0\ 0\ 0]^T$	$[0\ 0\ 1\ 0\ 0\ 0]^T$
$\underline{F}_1$	$[0\ 0.5\ 0\ 0\ 0\ 0.5]^T$	$[0\ 0\ 0.5\ 0\ -0.5\ 0]^T$
$\underline{F}_2$	$[0\ 0.5\ 0\ 0\ 0\ -0.5]^T$	$[0\ 0\ 0.5\ 0\ 0.5\ 0]^T$
	Case 3	Case 4
$\underline{P}$	$[0\ 1\ 0\ 0\ 1\ 0]^T$	$[0\ 0\ 1\ 0\ 0\ 1]^T$
$\underline{F}_1$	$[0\ 0.5\ 0\ 0\ 0.5\ 0.5]^T$	$[0\ 0\ 0.5\ 0\ -0.5\ 0.5]^T$
$\underline{F}_2$	$[0\ 0.5\ 0\ 0\ 0.5\ -0.5]^T$	$[0\ 0\ 0.5\ 0\ 0.5\ 0.5]^T$

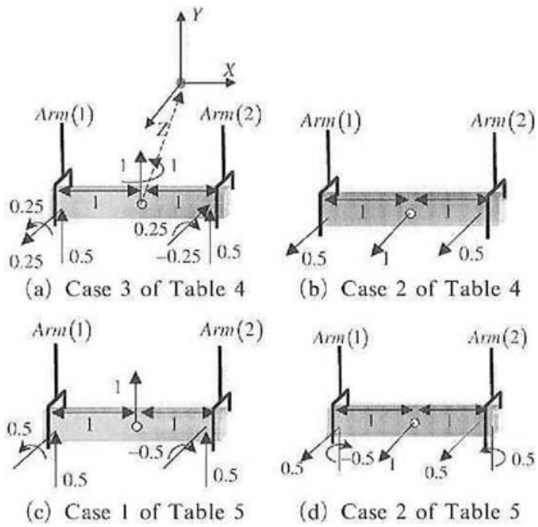


Fig. 12 Comparison of internal loadings for two cases

suggested by Walker is employed, it is found again that the internal loading exists as shown in Table 5. The Walker's algorithm is incorrect. In Fig. 12, we compare the case of internal loading with that of no internal loading.

4. Internal Force Analysis at the Three-Fingered and Three-Legged Systems

The previous examples were for tightly or rigidly grasped case. Now, we will consider a special interface between the grasped object and the multiple manipulators. We assume that each fingertip makes a point contact offering friction force to the object. Grasping space analysis is good for the analysis of this type of problems. In this case, Eq. (5) is transformed as

$$G=\begin{bmatrix} I_3 & I_3 & \cdots & I_3 \\ S_1 & S_2 & \cdots & S_n \end{bmatrix} \in R^{6\times 3m} \tag{67}$$

The first term on the right-hand side of Eq. (7) represents the manipulating force, and the second term denotes grasping force or internal force. Kumar et al. (1988) also defined those terms as an equilibrating force and an interaction force, respectively. The equilibrating forces are the forces required to maintain equilibrium against an external load, and the interaction force must have a zero net resultant. The definition of the interaction force is similar to the internal force. So, the number of the internal forces or the interaction force is $3\times n-6$[3, 4] if and only if the contact points are non co-linear. However, the interaction forces exist on the $(\underline{r}_i-\underline{r}_j)$ line, and the internal forces also exist in any direction at the coplanarity of contact points.

Yoshikawa et al. (1987) defined the internal loading by using three unit vectors given by

$$\underline{e}_1=\frac{\underline{r}_3-\underline{r}_2}{\|\underline{r}_3-\underline{r}_2\|},\ \underline{e}_2=\frac{\underline{r}_1-\underline{r}_3}{\|\underline{r}_1-\underline{r}_3\|},\ \underline{e}_3=\frac{\underline{r}_2-\underline{r}_1}{\|\underline{r}_2-\underline{r}_1\|} \tag{68}$$

where $\underline{e}_i$ is the unit vector directing from C_i to C_{i+1}. Then, the internal forces can be constructed by

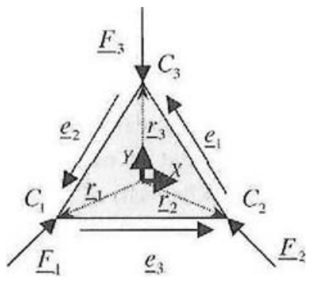


Fig. 13 Internal force

$$\begin{aligned}\underline{F_1}&=z_3\underline{e_3}+z_2(-\underline{e_2})\\ \underline{F_2}&=z_1\underline{e_1}+z_2(-\underline{e_3})\\ \underline{F_3}&=z_2\underline{e_2}+z_1(-\underline{e_1})\end{aligned} \tag{69}$$

where z_1, z_2, and z_3 are arbitrary real numbers. The matrix form of the internal force can be expressed as

$$\begin{bmatrix}\underline{F_1}\\ \underline{F_2}\\ \underline{F_3}\end{bmatrix}=\begin{bmatrix}0 & -\underline{e_2} & \underline{e_3}\\ \underline{e_1} & 0 & -\underline{e_3}\\ -\underline{e_1} & \underline{e_2} & 0\end{bmatrix}=\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} \tag{70}$$

and three internal forces form the closed triangle.

4.1 Three-fingered system

When a three-fingered system contacts a common object, the transformation matrix can be expressed as

$$G=\begin{bmatrix}I_3 & I_3 & I_3\\ S_1 & S_2 & S_3\end{bmatrix} \tag{71}$$

The position vectors of the contact point are $\underline{r_1}=[-1\ \ -\sqrt{3}/3\ \ 0]^T$, $\underline{r_2}=[1\ \ -\sqrt{3}/3\ \ 0]^T$, and $\underline{r_3}=[0\ \ 2\sqrt{3}/3\ \ 0]^T$, and the internal force matrix can be obtained as

$$I-G^+G$$
$$=\begin{bmatrix}0.58 & 0.14 & 0 & -0.42 & -0.14 & 0 & -0.17 & 0 & 0\\ 0.14 & 0.42 & 0 & 0.14 & -0.08 & 0 & -0.29 & -0.33 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -0.42 & 0.14 & 0 & 0.58 & -0.14 & 0 & -0.14 & 0 & 0\\ -0.14 & -0.08 & 0 & -0.14 & 0.42 & 0 & 0.29 & -0.33 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -0.17 & -0.29 & 0 & -0.17 & 0.29 & 0 & 0.33 & 0 & 0\\ 0 & -0.33 & 0 & 0 & -0.33 & 0 & 0 & 0.67 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \tag{72}$$

This matrix is a 9 by 9 square matrix whose rank is 3. Therefore, the basis of internal force is 3. Column 3, 6, and 9 of Eq. (72) have zero value. This is because the plane of the contact points is perpendicular to the Z-axis of the object coordinate. By using row-reduced echelon method, the basis of internal loading can be obtained as

$$[1\ 0\ 0\ 0\ -1.73\ 0\ -1\ 1.73\ 0]^T \tag{73}$$

$$[0\ 1\ 0\ 0\ 1\ 0\ 0\ -2\ 0]^T \tag{74}$$

$$[0\ 0\ 0\ 1\ -1.73\ 0\ -1\ 1.73\ 0]^T \tag{75}$$

Eqs. (73) through (75) form a closed force triangle. The interaction force can be obtained using the concept of force equilibrium at the contact point. From Eq. (70), the z vectors corresponding to the internal forces given in Eqs. (73) through (75) can be obtained as

$$z_1=[-2\ 0\ 1] \tag{76}$$

$$z_2=[-2/\sqrt{3}\ \ -2/\sqrt{3}\ \ 1/\sqrt{3}] \tag{77}$$

$$z_3=[-2\ 0\ 0] \tag{78}$$

Thus, the reduced row echelon method can be employed as a general approach to analyze the internal loading for general types of multiple robotic systems.

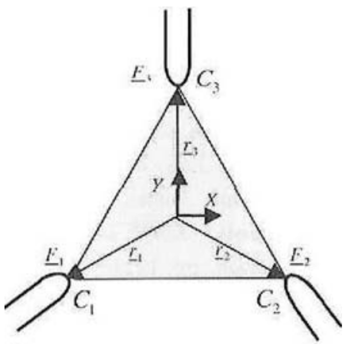


Fig. 14 Three-fingered system

4.2 Three-legged system

The concept of the internal force at the three-legged system is the same as that of three-fingered system, but the difference is the object being grasped. The position vectors of the end-point

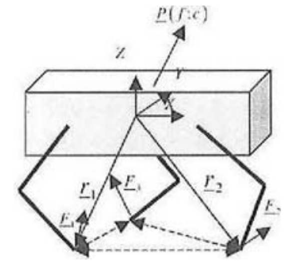


Fig. 15 Three-legged system

of legs are $\underline{r}_1=[-2\ \ -2\sqrt{3}/3\ \ -3]^T$, $\underline{r}_2=[2\ \ -2\sqrt{3}/3\ \ -4]^T$, and $\underline{r}_3=[0\ \ 4\sqrt{3}/3\ \ -4]^T$, and the internal force matrix can be obtained as

$$I-G^{+}G$$
$$=\begin{bmatrix} 0.54 & 0.13 & -0.15 & 0.39 & -0.13 & 0.12 & -0.15 & 0.01 & 0.04 \\ 0.13 & 0.39 & -0.09 & 0.14 & -0.08 & -0.02 & -0.27 & -0.32 & 0.11 \\ -0.15 & -0.09 & 0.05 & 0.08 & 0.04 & -0.03 & 0.08 & 0.04 & -0.03 \\ -0.39 & 0.14 & 0.08 & 0.56 & -0.15 & -0.12 & -0.17 & 0.01 & 0.04 \\ -0.13 & -0.08 & 0.04 & -0.15 & 0.41 & -0.02 & 0.28 & -0.34 & -0.02 \\ 0.17 & -0.02 & -0.03 & -0.12 & -0.02 & 0.03 & 0 & 0.05 & -0.01 \\ -0.15 & -0.27 & 0.08 & -0.17 & 0.28 & 0 & 0.32 & -0.01 & -0.08 \\ 0.01 & -0.32 & 0.04 & 0.01 & -0.34 & 0.05 & -0.01 & 0.65 & -0.09 \\ 0.04 & 0.11 & 0.03 & 0.04 & 0.02 & -0.01 & -0.08 & -0.09 & 0.03 \end{bmatrix} \quad (79)$$

This matrix is a 9 by 9 square matrix whose rank is 3. Therefore, the basis of internal force is 3, and by using row-reduced echelon, the internal forces can be obtained as

$$[1\ 0\ -0.25\ 0\ -1.73\ 0.25\ -1\ 1.73\ 0]^T \quad (80)$$

$$[0\ 1\ -0.14\ 0\ 1\ -0.14\ 0\ -2\ 0.28]^T \quad (81)$$

$$[0\ 0\ 0\ 1\ -1.73\ 0\ -1\ 1.73\ 0]^T \quad (82)$$

Note that Kumar et al.(1988) derived the condition for zero interaction force between legs. It is described as

$$\begin{aligned} (\underline{F}_{m1}-\underline{F}_{m2})\cdot(\underline{r}_1-\underline{r}_2)=0 \\ (\underline{F}_{m1}-\underline{F}_{m3})\cdot(\underline{r}_1-\underline{r}_3)=0 \\ (\underline{F}_{m2}-\underline{F}_{m3})\cdot(\underline{r}_2-\underline{r}_3)=0 \end{aligned} \quad (83)$$

where $\underline{F}_{mi}$ denotes the manipulating force at the i-th leg. The internal force vectors satisfy the condition for zero interaction force. Therefore, the reduced row echelon method is accepted as a general approach to analyze the internal loading of multiple robotic systems.

5. Conclusions

The internal loading can be defined as the forces and moments that do not affect the motion of the end-effecter. The basis of the internal loading has $6\times(n-1)$ independent vectors. However, there has been some confusion in using the concept of internal loading in multiple robotic systems such as multiple arms, multi-fingered hands, and walking machines. This paper aims at clarifying the physical meaning of internal loading by using the grasping space based analysis. A reduced row echelon method is employed to analyze the null space of multiple robotic systems. Firstly, the condition of no internal loading at the grasping space is derived, and is met when the weighting matrix is an identity matrix. Secondly, we analyze the internal loadings and show the shapes of the internal loading through various examples. Finally, the internal forces for three-fingered and three-legged systems are analyzed as special examples.

Acknowledgments

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